RESEARCH ARTICLE | JULY 06 2009

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AIP Conf. Proc. 1140, 60–68 (2009)
https://doi.org/10.1063/1.3183529

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Abstract. A key postulate of the causal set program is that this discrete partial order offers a sufficiently rich structure to make it a viable model of spacetime for quantum gravity. If the deep structure of spacetime is that of a causal set, then the correspondence principle (with the spacetimes of general relativity) must be obeyed. Therefore, one of the requirements of this program is to establish that the causal set structure is in fact, not just in principle, fully consistent with our macroscopic notion of spacetime as a Lorentzian manifold. An important component of any such "manifold test" is the ability to find embeddings of causal sets into Lorentzian manifolds.

Keywords: quantum gravity, discrete spacetime, causal sets
PACS: 04.60.-m, 04.60.Nc

INTRODUCTION

Quantum gravity continues to be one of the most interesting and elusive problems in theoretical physics. The two most popular approaches that seek to solve this problem are string theory and loop quantum gravity. However, there are many other approaches and among these is causal sets. The causal set proposal started in the late 1970s [1, 2, 3] and began in earnest with the seminal paper by Bombelli *et. al.* [4]. See references [5]-[9] for general introductions to causal sets.

A causal set is a set C of elements $x_i \in C$, and an order relation $\prec$, such that $C = \{x_i, \prec\}$ obeys properties which make it a good discrete counterpart for continuum spacetime. These properties are:

1. *transitivity*: if $x_i \prec x_j \prec x_k \Rightarrow x_i \prec x_k$;
2. *finitarity*: the number of elements between any two ordered elements $x_i \prec x_j$ is finite, that is, $\left|[x_i, x_j]\right| < \infty$;
3. *noncircularity*: if $x_i \prec x_j$ and $x_j \prec x_i \Rightarrow x_i = x_j$
4. *reflexivity*: $x_i \prec x_i \; \forall \; x \in C$.

Transitivity and noncircularity say that this structure is a partially ordered set, or *poset* for short. Specifically, non-circularity amounts to the exclusion of closed timelike curves; this condition may be relaxed in some models. The finitarity of the set ensures that it is discrete. The reflexivity requirement is present as a convenience to eliminate the ambiguity of how an event relates to itself. We can combine these statements to give the following definition: *A causal set is a locally finite, partially ordered set.*

CP1140, *Proceedings of the National Society of Black Physicists*, edited by H. M. Oluseyi

THE IMPORTANCE OF EMBEDDING

Order + Number $\Rightarrow$ Geometry

As discussed in any of the general references on causal sets, one of the foundational ideas of this approach is the fact that the causal structure of a spacetime determines almost all of the information needed to specify the metric [10, 11] and therefore the gravitational field tensor. The causal structure determines the metric up to an overall multiplicative function called a conformal factor. Two metrics $g_{\mu\nu}$ and $\widetilde{g}_{\mu\nu}$ are conformally equivalent if $\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where Ω is a smooth positive function. Since all conformally equivalent spacetimes have the same causal structure [12] the causal structure itself nearly specifies the metric.

Lacking the conformal factor means that we lack the size scale of the metric that allows for quantitative measures of lengths and volumes. However, if spacetime is discrete, the volume of a region can be determined from the number of events within that region. Thus, the combination of causal structure (order) and discreteness (number) provides, in principle, enough information to construct the spacetime metric (geometry). There have been many investigations related to the recovery of geometric information from the structure of a causal set. These investigation include determining the dimension of the corresponding Lorentzian manifold [13, 14], the recovery of geodesics [15, 16, 17], and topology [18].

The Main Conjecture of Causal Set Kinematics

Another foundational principle at the core of the development of causal sets is what has been called its main conjecture, termed the *Hauptvermutung* [6, 19], which effectively says that, in the appropriate limits, a physically interesting causal set will bring forth all of the structure of a nearly unique spacetime manifold. Indeed, this sort of idea must be true if a correspondence between causal sets and macroscopic spacetimes is to be established.

However, this main conjecture, as stated above, begs the following question: What determines whether a causal set is physically interesting? The answer, which applies to both the `order + number ⇒ geometry` principle and its underlying *Hauptvermutung*, is that physically interesting causal sets are those that are *faithfully embeddable* into Lorentzian manifolds.

Faithful Embeddings

An embedding of a causal set is a mapping of the set onto points in a Lorentzian manifold such that the lightcone structure of the manifold preserves the ordering of the set. The most probable embeddings will be uniform if the mapping corresponds to selecting points in the manifold via a Poisson process. When this latter requirement is met, the number of causal set elements mapped into any region of the manifold is directly proportional to the volume of that region. Under these circumstances, the embedding is said to be *faithful*.

In practice, to construct an embedding you need a causal set C, and a manifold $\mathscr{M}$ with a metric g of Lorentzian signature. Choose a coordinate system $\Psi(\mathscr{M})$ for points in $\mathscr{M}$ and assign points to elements in the causal set by performing a mapping $\psi : C \rightarrow \Psi(\mathscr{M})$ such that the causal structure induced by the lightcones of g on the images of $\psi(C)$ preserves the ordering of the corresponding elements of C. Ideally, the algorithm by which the mapping ψ is performed produces an embedding $\psi(C)$ that is faithful. This means that, with high probability, the embedding $\psi(C)$ can be reproduced by randomly selecting points in $\Psi(\mathscr{M})$ according to the Poisson distribution.

The key point here is that *everything* we assume about the correspondence between causal sets and classical gravity, which is the foundation on which the whole program is built, rests upon the following idea: *A dynamical process exists that will produce physically meaningful, faithfully embeddable causal sets. Faithful embeddings of these sets can be found and used to extract physical information concerning the properties of spacetime.* Without a practical way to embed causal sets, we may never be able to place the approach on truly firm mathematical ground and may not even be able to find ways to meaningfully use causal sets as a physical model for spacetime.

Sprinkling

As stated above, we are most interested in embeddings that could correspond to selecting points in a manifold according to the Poisson distribution. Actually performing this selection is an excellent way to produce a faithfully embeddable causal set to use as a testing ground for embedding algorithms and other studies – we call this process a *random sprinkling* of points. There are a number of ways to produce a faithfully embeddable causal set by sprinkling. Since the present results below are restricted to 1+1-dimensional Minkowski space, a very simple method can be used.

It is more useful to study causal set *intervals* where an interval between two related elements $I_C[x,z]$ is the inclusive subset $I_C[x,z] = \{y_i | x \prec y_i \prec z\}$. If we think of $\prec$ as a causal order, $I_C[x,z]$ is the intersection of the future of x with the past of z. A Poisson sprinkling of N points into an interval of two-dimensional Minkowski space is easily obtained by randomly selected points in a square region defined by an origin in the lower-left corner $(u,v) = (0,0)$ and the upper-right corner $(u,v) = (a,a)$, then rotating that region by 45°. This rotation forms a spacetime interval $I_{\mathscr{M}}[x,z]$ with the lower-left corner as the pastmost point x and the upper-right corner as the futuremost point z. This process is illustrated in Figure 1.

To turn this sprinkling into a faithfully embeddable causal set, use the lightcones from the selected points to determine which events are causally ordered and keep track of this information. Next, remove the manifold so that the points are no longer associated with any coordinates. Then relate those points that were causally ordered in the manifold with the order relation $\prec$ and you are left with a bare causal set – one that is known to be faithfully embeddable.

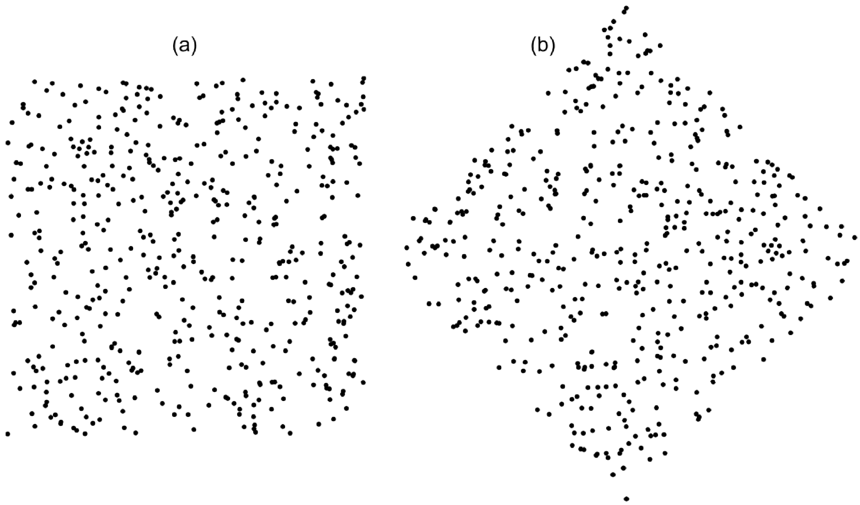

FIGURE 1. A Poisson sprinkling of 500 points into an interval of 1+1-dimensional Minkowski space. Panel (a) shows the points in (u,v) coordinates and panel (b) shows them after rotation to (t,x) coordinates.

THE STATUS OF RESEARCH ON EMBEDDING CAUSAL SETS

The study of embeddings of causal sets goes back to the early days of causal set research. The first embedding algorithm on record appears to be that of Bombelli and Meyer, see the Appendix of reference [19]. They associated an "energy" with the configuration of causal set elements and used simulated annealing in an attempt to find the minimum energy configuration. This configuration would be associated with an accurate embedding of the causal set. Their results were that this algorithm was successful at embedding some simple, small causal sets, but could not find embeddings of more complicated causal sets, even if they were known to be embeddable.

Meyer [13] and Daughton [20] have studied the symmetric embeddability of the binomial poset. The binomial poset on N elements, B_N, contains 2^N elements layered such that the mth layer contains $\binom{N}{m}$ elements. It consists of all subsets of the N elements, including the null set. The structure of B_N is such that none of the elements on the mth layer are related to each other, while every element on this layer is related to every element on layer $m+1$. The importance of this structure is that every partially ordered set is a subset of *some* binomial poset [13].

Meyer showed that symmetric embeddings of B_N do not exist in any Minkowski space for $N \geq 6$. Daughton extended Meyer's work to consider curved manifolds with topology $\mathbb{S}^{N-1} \times \mathbb{R}$; he found that no symmetric embeddings exist for $N \geq 7$. These results can be useful in many contexts. Their importance here is that one might use these facts, as a part of a larger embedding algorithm, to make quick decisions on whether a particular causal set is likely to embed into a particular spacetime. Given the very large configuration space that one has to search, such checks could have significant practical value.

For example, you can imagine a dynamics that generates a causal set for which you have no idea *a priori* if the causal set is embeddable and, if so, in what spacetime. Since we know that the middle two layers of B_3 is not embeddable into 1+1-dimensional Minkowski space but is in 2+1 dimensions [13], one can sample small regions of the causal set to look for this poset; if found, this suggests that it is better to abandon 1+1 dimensions in favor of searching for embeddings in 2+1-dimensional spacetimes.

Recently, Henson has developed an algorithm for embedding causal sets into Minkowski space which he has tested in 1+1 dimensions [21]. He used a two-step process the first of which was to employ a well-known estimator of the timelike distance between related elements to set coordinates for all the elements. This does not produce an exact embedding in that some of the relations between elements in the causal set do not match the causal relations between their images in the spacetime. In an attempt to make the embedding exact, a second step in which the previously selected coordinates are shifted was used. While he was not able to find an exact embedding with this scheme, he was able to match 99.66% of the relations on average for 10,000-element causal sets obtained from sprinklings.

As is evident from the above discussion, no one has yet developed an embedding algorithm that successfully produces exact embeddings of reasonably complicated causal sets. While accomplishing exact embeddings remains a goal, it is worth noting that one does not expect to need truly exact embeddings at the level of fundamental theory.

If the deep structure of spacetime is that of a causal set, then on their natural size scale, perhaps the Planck scale or smaller, one does not expect to have anything like a manifold. Trying to discern a manifold on this scale is like trying to read a computer screen on a scale that resolves the individual pixels that make up the letters. To discern the geometric structure of the letters we look at them at a significantly different scale. Similarly, we should require a mathematical change-of-scale to precisely associate a causal set with a particular manifold. This change-of-scale is called *coarse-graining*. Strictly speaking, it is a coarse grained causal set that one expects to embed into Lorentzian manifolds rather than every individual element of the causal set itself.

However, it is entirely likely that, even after coarse graining, a causal set dynamics will impose a stochastic character on the correspondence between causal sets and Lorentzian manifolds. Indeed, this character is crucial if we hope to preserve local Lorentz invariance [22]. Thus, we must allow for a fluctuating number of causal set elements – it is the *expected* number of elements $<N>$ that corresponds to volume. Given the fact that adding or subtracting elements from an embeddable causal set can destroy its embeddability, providing for these fluctuations seems to imply that we should not require exact embeddability of all causal set elements.

A SIMPLE SEQUENTIAL SEARCH ALGORITHM: PRELIMINARY RESULTS

The present calculations revive previously unpublished work along these lines and represents the first steps in what will likely be a long-term effort to find quick and effective algorithms to embed causal sets into Lorentzian manifolds. In its present form, the code employs an algorithm that takes causal sets formed by sprinkling into an interval of 1+1-dimensional Minkowski space and attempts to embed these elements back into an interval of 1+1-dimensional Minkowski space. The algorithm considers each element of a causal set in sequence and searches for valid coordinates on which to embed the element.

The most important step in constructing the embedding is selecting the time coordinate for an element. As in [21], the present calculations use the timelike distance estimator for this, but in a different way. To explain this estimator, let us first define a few terms. A *link*, $\preceq$, in a causal set is an irreducible relation; so, $x_i \preceq x_k$ iff $\nexists\, x_j \ni x_i \prec x_j \prec x_k$. A *chain* in a causal set is a set of elements for which each pair is related; for example, $x_a \prec x_b \prec \cdots \prec x_{z-1} \prec x_z$ is a chain from x_a to x_z. A *maximal chain*, or *path*, is a chain consisting only of links, such as $x_a \preceq x_b \preceq \cdots \preceq x_{z-1} \preceq x_z$. Recall that the length of the geodesic between two causally related events gives the longest proper time between those events. It appears to be Myrheim [2] who first argued that the length of the longest maximal chain between two related elements in a causal set is the most natural analog for the geodesic length between two causally connected events in spacetime; where the length of a path is taken to be the number of links it contains. The proof of this result is discussed in [17]. A fast algorithm for finding this length, used in the present calculations, has been provided by Sorkin [23].

In the present calculations, faithfully embeddable causal set intervals are obtained by sprinkling in 1+1-dimensional Minkowski space as previously described. The elements are labelled $1...N$ according to their time coordinates from the sprinklings. The minimal (x_1) and maximal (x_N) elements are embedded at $x_1 = (0,0)$ and $x_N = (T,0)$ where T is determined by the rotation of the upper-right corner of the sprinkled interval in uv coordinates $T = a\sqrt{2}$. Each of the remaining elements are considered sequentially in order of their labelling.

A time coordinate of each $x_2...x_{N-1}$ is obtained by determining the minimum time coordinate this element must have to be in the future of all previously embedded elements in its past and comparing this value to its "ideal" time coordinate as determined by the length of the longest path from x_1 to x_N containing that element. For example, if the length of the longest path that contains x_{32} is 15, and x_{32} is the 7th element in this path, the ideal time coordinate for x_{32} is $6T/15$. The time coordinate is chosen randomly near the ideal time if this time is greater than the minimum, or chosen randomly above the minimum if the minimum is greater.

Once a time coordinate is chosen, the lightcones of all previously embedded elements are projected onto that time slice. This spacelike surface is searched for regions in which the point can embed with the proper relation to all previously embedded elements. If multiple regions are found, one is selected randomly. Once a region is selected, the element is embedded randomly within it. In the present version of the code, if no valid region is found on the selected time slice, the element is discarded. The choice to discard elements that won't embed is made purely for convenience. One can certainly choose to embed the elements anyway and use the result as an *ansatz* into a refinement procedure as in [4, 8] or to study how close two causal sets must be, the one from the sprinkling and the one from the embedding, in order to have similar geometric features. It is almost certain that some studies along these lines will be performed in the future.

Table 1 summarizes the results of the present calculations. For each sprinkling of

TABLE 1. The number of elements successfully embedded for each sprinkling size N.

N	Number Embedded	N	Number Embedded
10	10	200	127
20	20	300	179
30	30	400	248
40	40	500	265
50	46	600	323
60	56	700	371
70	58	800	387
80	65	900	455
90	68	1000	498
100	79		

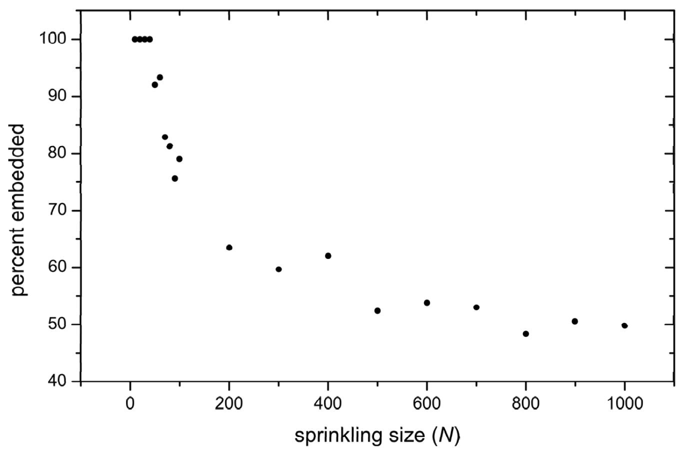

FIGURE 2. The percentage of points successfully embedded as a function of sprinkling size.

N elements, the code performs up to 10,000 trial embeddings of the causal set for $10 \leq N \leq 500$, up to 5000 trials for $500 \leq N \leq 800$, and 500 trials for $900 \leq N \leq 1000$. The results reported in table 1 represent the largest numbers of points successfully embedded for each sprinkling size. For N equal to 10, 20, 30, and 40 the number of trials needed to achieve these exact embeddings were 3, 2, 87, and 341 respectively. Even though the present calculations used only small causal sets, making it difficult to draw conclusions, the fact that about half (or more) of the elements are embedded is quite promising if one considers any level of coarse-graining to be necessary.

Figure 2 shows the percentage of points embedded versus sprinkling size. The trend of the data does hint at a flattening off somewhere between 40 - 50%. Perhaps this percentage range represents the best that the current scheme can do once the causal set reaches a certain degree of complexity. This will become clearer with further study.

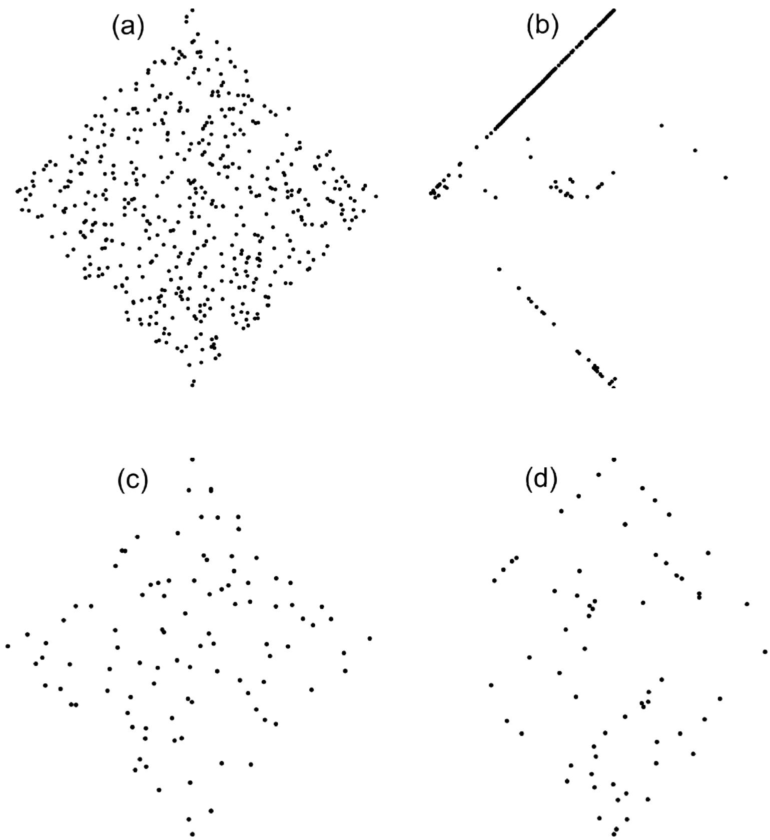

FIGURE 3. Representative examples of Poisson sprinklings and their embeddings. Panel (a) shows the result of sprinkling 500 points in an interval of Minkowski space and panel (b) is the resulting embedding of 262 of those points. Panel (c) is a sprinkling 100 points with panel (d) being an embedding of 66 of those points.

One clear drawback of the current algorithm is that, other than distributing the points in time, little is done to make the embedding faithful. As a result, some of the embeddings appear to have virtually no chance at being made into a faithful embedding in a second refinement stage for example, see panel (b) of figure 3, while other embeddings seem to have a chance such as in panel (d) of figure 3. A true measure of just how faithful an embedding is must be performed statistically.

CONCLUDING REMARKS

Results like those of Henson [21] and the present preliminary work seem to suggest that the goal of finding practical ways to embed causal sets into Lorentzian manifolds is realistic and within reach. Many refinements to the work presented here are possible. However, the most likely fate of any sequential search algorithm is as an initial configuration used as input for a better overall scheme. In the coming years, I hope to explore any number of avenues in pursuit of this goal including refinements of all the methods employed so far. Perhaps the most intriguing idea is to use some of the many techniques from information science, particularly cryptanalysis, given the significant similarities between the search for an embedding and codebreaking.

ACKNOWLEDGMENTS

I'd like to thank Chanda Prescod-Winstein for encouraging me to make this contribution.

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